Symmetry breaking and double quotient spaces

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1. Symmetry breaking

Given two groups $G' \subset G$, and an irreducible representation (π, V) of G, we ask the following question :

What can be said about $\pi_{|G'}$, and furthermore, about the sets of symmetry breaking operators $\mathrm{Hom}_{G'}(\pi_{|G'}, \tau)$ for some $\tau \in \mathrm{Irr}(G')$?

In this context, the Rankin-Cohen bidifferential operators $\mathcal{RC}_{\lambda,\mu}^{\lambda+\mu+2a}$ defined by

$$\mathcal{RC}_{\lambda,\mu}^{\lambda+\mu+2a}(f,g) = \sum_{r+s=0}^{a} (-1)^r \binom{\lambda+a-1}{s}\binom{\mu+a-1}{r} \frac{\mathrm{d}^r f}{\mathrm{d}z^r}\frac{\mathrm{d}^s g}{\mathrm{d}z^s}, \qquad \lambda, \mu \in \{2,3,\dots\}$$

may be interpreted as symmetry breaking operators for tensor products of discrete series representations π_λ and π_μ of $\mathrm{SL}(2,\mathbb{R})$. The Rankin-Cohen brackets make explicit the decomposition

$$\pi_\lambda \otimes \pi_\mu \simeq \bigoplus_{a \geqslant 0} \pi_{\lambda+\mu+2a}$$

when π_λ is realised on the Bergman space $\mathcal{O}(\mathcal{H}) \cap L^2(\mathcal{H}, y^{\lambda-2}\,\mathrm{d}x\,\mathrm{d}y)$.

2. Principal series representations

G : reductive Lie group, $\quad P$: parabolic subgroup, $\quad (\chi, V)$: fin. dim. rep. of P.

We define $\mathcal{V} := (G \times V)/\sim$ where $(g,v) \sim (gp, \chi(p)^{-1}v)$.
$\mathcal{V}$ is an **homogeneous vector bundle** over G/P and the space of smooth sections $C^\infty(G/P, \mathcal{V})$ is a representation space of G once endowed with the natural Fréchet topology and the action defined by

$$\begin{tikzcd} \mathcal{V} \arrow[d, "p"'] \\ G/P \arrow[u, bend right, dotted, "s"'] \end{tikzcd}$$

$$\forall g \in G, \forall s \in C^\infty(G/P, \mathcal{V}), \forall m \in G/P, \quad \pi(g)s(m) = gs(g^{-1}m).$$

3. Distribution kernel of a symmetry breaking operator

$$\begin{array}{lll} G' \subset G : & \text{reductive Lie groups,} & (\tau, W), (\chi, V) : \text{fin. dim. rep. of } P' \text{ and } P, \\ \cup \quad\ \cup & & \\ P' \quad\ P : & \text{parabolic subgroups,} & \mathcal{W} \to G'/P',\ \mathcal{V} \to G/P : \text{hom. vec. bundles.} \end{array}$$

We define the **dualizing bundle** $\mathcal{V}^* := \mathcal{V}^\vee \otimes \Omega_{2\rho}$ where $\Omega_{2\rho}$ is the density bundle of G/P.
Distributions are involved because of Schwartz's Kernel Theorem (see [Sch57], Proposition 25).

Theorem. *There is an isomorphism*

$$\mathrm{Hom}_{\mathbb{R}}(C^\infty(G/P, \mathcal{V}), C^\infty(G'/P', \mathcal{W}) \simeq \mathcal{D}'(G/P \times G'/P', \mathcal{V}^* \boxtimes \mathcal{W}).$$

Its importance for branching problems is due to the following result (see [KP16], Lemma 2.21).

Fact. *The following isomorphisms hold*

$$\begin{aligned} \mathrm{Hom}_{G'}(C^\infty(G/P, \mathcal{V}), C^\infty(G'/P', \mathcal{W})) &\simeq \mathcal{D}'(G/P \times G'/P', \mathcal{V}^* \boxtimes \mathcal{W})^{\mathrm{diag}(G')} \\ &\simeq \Big(\mathcal{D}'(G/P, \mathcal{V}^*) \otimes W\Big)^{\mathrm{diag}(P')}. \end{aligned}$$

Observation. If $T \in \mathrm{Hom}_{G'}(C^\infty(G/P, \mathcal{V}), C^\infty(G'/P', \mathcal{W}))$, the **support** of its distribution kernel $K_T \in \Big(\mathcal{D}'(G/P, \mathcal{V}^*) \otimes W\Big)^{\mathrm{diag}(P')}$ is a **closed P'-invariant subset** of G/P.

Idea : Analysis of $P'\backslash G/P$ provides an information on K_T, hence on T.

4. The Grassmannian of k-planes

We consider the groups

$$G = \mathrm{GL}(n,\mathbb{R}) \supset G' = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{GL}(n-1,\mathbb{R}) \end{pmatrix}, \quad P = \begin{pmatrix} \mathrm{GL}(k,\mathbb{R}) & \mathrm{M}_{k,n-k}(\mathbb{R}) \\ 0 & \mathrm{GL}(n-k,\mathbb{R}) \end{pmatrix} \subset G, \ P' = G' \cap P$$

and use the identifications

$$\begin{aligned} G/P \simeq \mathrm{Gr}_k(\mathbb{R}^n) &:= \{k\text{-dim. subspace of } \mathbb{R}^n\} \\ &\simeq \{A \in \mathrm{M}_{n,k}(\mathbb{R}) \mid \mathrm{rank}\, A = k\}/\mathrm{GL}(k,\mathbb{R}). \end{aligned}$$

If $J \subset \{1, \dots, n\}$ is non-empty, the map

$$\begin{array}{ccc} \mathrm{M}_{n,k}(\mathbb{R}) & \to & \mathbb{N} \\ (a_{ij}) & \mapsto & \mathrm{rank}\Big((a_{ij})_{(i,j) \in J \times \llbracket 1,k \rrbracket}\Big) \end{array}$$

induces a **well-defined lower semi-continuous map** $\mathrm{rank}_J : \mathrm{Gr}_k(\mathbb{R}^n) \to \mathbb{N}$.
It is P'-invariant for $J = \{1\}$, $\llbracket 2,n \rrbracket$, $\llbracket k+1,n \rrbracket$ and $\{1\} \cup \llbracket k+1,n \rrbracket$.

5. Classification of P'-orbits on G/P

Theorem A. *Set $s = \min(k, n-k)$. The P'-orbits on $\mathrm{Gr}_k(\mathbb{R}^n)$ are*

$$\underset{1 \leqslant r \leqslant s}{X_r^1} = \big\{W \in \mathrm{Gr}_k(\mathbb{R}^n) \mid \mathrm{rank}_{\{1\}} W = 0,\ \mathrm{rank}_{\llbracket k+1,n \rrbracket} W = r\big\},$$

$$\underset{1 \leqslant r \leqslant s}{X_r^2} = \left\{W \in \mathrm{Gr}_k(\mathbb{R}^n) \,\middle|\, \begin{array}{c} \mathrm{rank}_{\{1\}} W = 1,\ \mathrm{rank}_{\llbracket k+1,n \rrbracket} W = r, \\ \mathrm{rank}_{\{1\} \cup \llbracket k+1,n \rrbracket} W = r \end{array}\right\},$$

$$\underset{2 \leqslant r \leqslant s}{X_r^3} = \left\{W \in \mathrm{Gr}_k(\mathbb{R}^n) \,\middle|\, \begin{array}{c} \mathrm{rank}_{\{1\}} W = 1, \mathrm{rank}_{\{1\} \cup \llbracket k+1,n \rrbracket} W = r \\ \mathrm{rank}_{\llbracket 2,n \rrbracket} W = k,\ \mathrm{rank}_{\llbracket k+1,n \rrbracket} W = r-1 \end{array}\right\},$$

$$\underset{1 \leqslant r \leqslant s}{X_r^4} = \big\{W \in \mathrm{Gr}_k(\mathbb{R}^n) \mid \mathrm{rank}_{\llbracket 2,n \rrbracket} W = k-1,\ \mathrm{rank}_{\llbracket k+1,n \rrbracket} W = r-1\big\}.$$

Moreover, these are embedded submanifolds of dimension

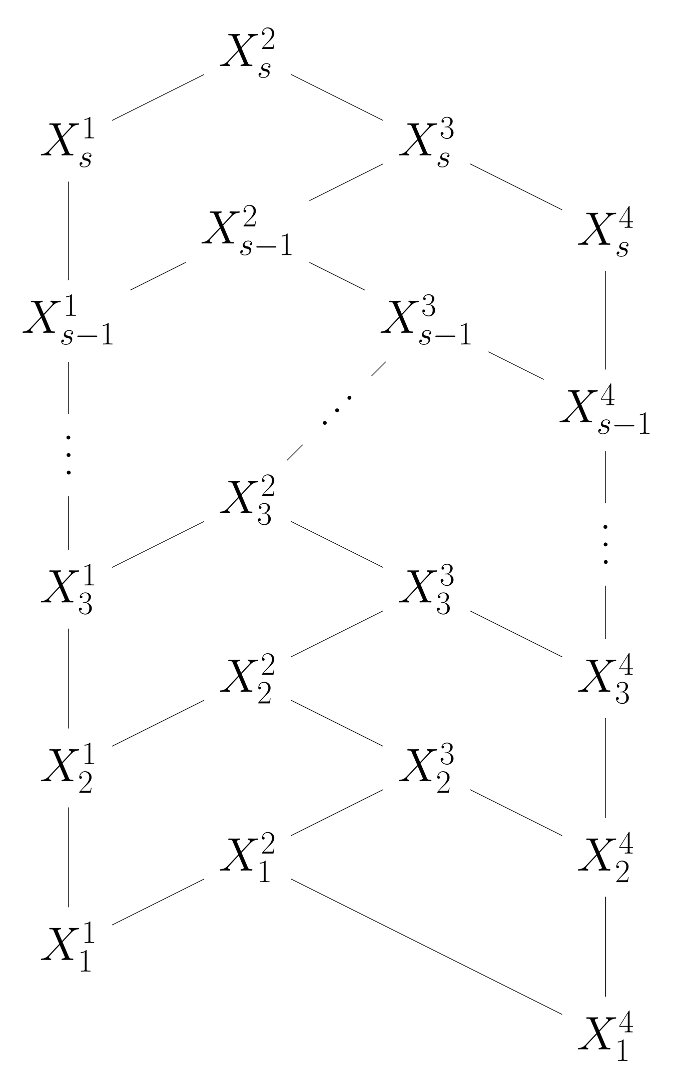
$$\begin{array}{ll} \dim X_r^1 = r(n-r) - k, & \dim X_r^2 = r(n-r+1) - k, \\ \dim X_r^3 = (r-1)(n-r+1), & \dim X_r^4 = (r-1)(n-r). \end{array}$$

6. Hasse diagram

If $\mathcal{O} \subset G/P$ is a P'-orbit, its closure $\overline{\mathcal{O}}$ in G/P decomposes uniquely as a disjoint union of P'-orbits. We define a binary relation $\prec$ on $P'\backslash G/P$ as follows :

$$\forall\, \mathcal{O}, \mathcal{O}' \in P'\backslash G/P,\ \mathcal{O} \prec \mathcal{O}' \iff \mathcal{O} \subset \overline{\mathcal{O}'}.$$

Theorem B. *$(P'\backslash G/P, \prec)$ is a poset whose Hasse diagram is the following :*



References

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